RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. FOURTH SEMESTER EXAMINATION, MAY 2018

SECOND YEAR (BATCH 2016-19)

MATHEMATICS (Honours)

Date : 19/05/2018

Time : 11.00 am – 3.00 pm

Paper : IV

Full Marks : 100

[Use a separate Answer Book for each group]

Group - A

Answer any five questions from Question No. 1 to 8 : [5×7]

1. a) Let (X, d) be a metric space, $A \subseteq X$ and $x \in X$. Prove that $x \in \bar{A}$ iff d(x, A) = 0.
   b) Suppose $A \subseteq \mathbb{R}$. Show that the set of all isolated points of A is at most a countable set. [3+4]

2. a) Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence in a metric space X converging to $x_0 \in X$. Prove that $\{x_n\}_1^{\infty} \cup \{x_0\}$ is compact.
   b) Define a Baire space. Show that with usual metric, the space $\mathbb{Q}$ is not a Baire space. [3+4]

3. Let (X,d) be a metric space such that $d(A,B) > 0$ for any two disjoint closed subsets A, B of X. Show that (X,d) is complete. Is the converse true? Justify your answer. [4+3]

4. a) Let (X,d) be a metric space and A, B are compact subsets of X. Show that there exists $a \in A$ and $b \in B$ such that d(A,B) = d(a,b) and if A and B are disjoint then $d(A,B) > 0$.
   [d(A, B) = inf {d(x, y): $x \in A$, $y \in B$}]
   b) Let X and Y be metric spaces and there exists a continuous map $f : X \to Y$ such that $G(f) = \{(x, f(x)) : x \in X\}$ is not a complete subset of X × Y. Prove that either X or Y fails to be a complete metric space. [4+3]

5. a) Suppose X is a metric space such that every uncountable set in X has a limit point. Show that X is separable.
   (Hint : For each $n \in \mathbb{N}$, by Zorn's lemma, choose a maximal set in X, the distance between any two points of which is at least $\frac{1}{n}$).
   b) Suppose 'd' is a metric on $\mathbb{N}$. Show that $(\mathbb{N}, d)$ is 2nd countable. [5+2]

6. a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a polynomial. Show that f sends every closed set in $\mathbb{R}$ to a closed set in $\mathbb{R}$.
   b) Let $A \subseteq \mathbb{R}$ be such that each continuous map $f : A \to \mathbb{R}$ is bounded. Show that A is compact. [4+3]

7. a) Let C[0,1] be the set of all real valued continuous maps over [0,1]. Define two metrices d_1 and d_2 on C[0,1] as follows :

   $$d_1(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)| \,;\; d_2(f,g) = \left(\int_0^1 (f-g)^2 dx\right)^{\frac{1}{2}}$$

   Check whether d_1 and d_2 are equivalent or not.
   b) Give example of a connected subset of a metric space such that interior of the set is not connected. [4+3]

8. a) Show that $\mathbb{R}$ is connected.

b) Let $X = \mathbb{N} \cup \{a\}$ where $a \notin \mathbb{N}$. Define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = 1,\ x, y \in \mathbb{N},\ x \neq y$$

$$d(a, x) = d(x, a) = 1 + \frac{1}{x}, x \in \mathbb{N}$$

$$d(x, y) = 0,\ x = y;\ x, y \in X$$

Show that (X,d) is a complete metric space. [3+4]

Answer <u>any three</u> questions from <u>Question No. 9 to 13</u> : [3×5]

9. a) Prove that if a power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = x_1$ and $x = x_2$, then the power series converges for any x between x_1 and x_2.

b) Let f be defined on $\left(-\frac{1}{3}, \frac{1}{3}\right)$ by $f(x) = 1 + 2\cdot 3x + 3\cdot 3^2 x^2 + \ldots + n \cdot 3^{n-1} x^{n-1} + \ldots$. Prove that f is continuous on $\left(-\frac{1}{3}, \frac{1}{3}\right)$. [3+2]

10. Show that the series $\sum_{n=1}^{\infty} \frac{x}{(nx+1)\{(n-1)x+1\}}$ is uniformly convergent on [a,b] for 0 < a < b. Justify whether it is uniformly convergent in [0,b]. [4+1]

11. Check the convergence and uniform convergence of the following sequence of functions

a) $\frac{nx}{1+n^3x^2}$, $n \in \mathbb{N},\ x \in [0,1]$

b) nxe^{-nx^2}, $x \geq 0, n \in \mathbb{N}$. [3+2]

12. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$ is uniformly convergent over $\mathbb{R}$. If S(x) is the sum function, verify that $S'(x)$ is obtained by term by term differentiation. [2+3]

13. Prove that a power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R > 0 iff $\sum_{n=0}^{\infty} na_n x^{n-1}$ has the same radius of convergence.

<u>Group - B</u>

Answer <u>any three</u> questions from <u>Question No. 14 to 18</u> : [3×10]

14. a) Solve the equation $(y^2 + z^2 - x^2)dx - 2xydy - 2zxdz = 0$ after satisfying the condition of integrability.

b) Find the solution in series of the equation $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2 y = 0$ about x = 0. [5+5]

15. a) Solve $(1-x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x(1-x^2)$ given that y = x is a solution of its reduced equation.

b) Solve : $\frac{dx}{x^2+a^2} = \frac{dy}{xy-az} = \frac{dz}{xz+ay}$.

c) Use convolution theorem to find $L^{-1}\left\{\frac{1}{(p+1)(p-2)}\right\}$. [4+4+2]

16. a) Find the complete integral of the partial differential equation $z^2 = \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}xy$ by Charpit's method.

b) Solve the equation : $(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = (x-1)^2$ using the method of variation of parameters. [5+5]

17. a) Find all the eigen-values and eigen-functions of $\frac{d}{dx}\left(4e^{-x}\frac{dy}{dx}\right) + (1+\lambda)e^{-x}y = 0$; $y(0) = 0$, $y(1) = 0$.

b) Find the equation of the integral surface given by the differential equation $2y(z-3)p + (2x-z)q = y(2x-3)$, which passes through the circle $z = 0$, $x^2 + y^2 = 2x$. [5+5]

18. a) Determine the solution of the following initial value problem by Laplace transform technique : $(D^2 + 2D + 1)x = 3t\,e^{-t}$, given $x(0) = 4$, $x'(0) = 2$.

b) If F(t) be a periodic function with period T > 0, then prove that $L\{F(t)\} = \frac{\int_0^T e^{-pt}F(t)dt}{1-e^{-pT}}$.

c) Solve : $x\frac{d^2y}{dx^2} + (x-1)\frac{dy}{dx} - y = x^2$, by the method of operational factors. [4+3+3]

Answer <u>any two</u> questions from <u>Question No. 19 to 21</u> : [2×10]

19. a) Show that the points on the curve $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$ where the tangents are parallel to x-axis lie on the parabola $y^2 = 4ax$.

b) Find the pedal equation of the equiangular spiral $r = ae^{\theta\cot\alpha}$.

c) Show that the radius of curvature at any point on the Cardiode $r = a(1-\cos\theta)$ is proporsonal to $\sqrt{r}$. [4+3+3]

20. a) Find the envelope of $\frac{x}{a} + \frac{y}{b} = 1$ with the condition $a^n + b^n = c^n$, where ‘a’ and ‘b’ are parameters and ‘n’ and ‘c’ are real constants.

b) Find the asymptotes of the curve $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0$. [5+5]

21. a) Find the area of the surface generated by revolving about the y-axis that part of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$, that lies in the first quadrant.

b) Determine the position and nature of the multiple point(s) of the curve $x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0$.

c) Show that $y = x^4$ is concave upwards at the origin and $y = e^x$ is everywhere concave upwards. [4+4+2]

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